ISOMORPHY CLASSES OF FINITE ORDER AUTOMORPHISMS OF $\mathrm{SL}(2,k)$

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Abstract. In this paper, we consider the order m k-automorphisms of $\mathrm{SL}(2,k)$. We first characterize the forms that order m k-automorphisms of $\mathrm{SL}(2,k)$ take and then we simple conditions on matrices A and B, involving eigenvalues and the field that the entries of A and B lie in, that are equivalent to isomorphy between the order m k-automorphisms Inn_A and Inn_B. We examine the number of isomorphy classes and conclude with examples for selected fields.

1. Introduction

Let G be a connected reductive algebraic group defined over a field k of characteristic not two, ϑ an involution of G defined over k, H a k-open subgroup of the fixed point group of ϑ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. For $k = \mathbb{R}$ these symmetric k-varieties are also called real reductive symmetric spaces. These varieties occur in many problems in representation theory, geometry and singularity theory. To study these symmetric k-varieties one needs first a classification of the related k-involutions. A characterization of the isomorphism classes of k-involutions was given in [Hel00].

In [HW02], a full characterization of the isomorphism classes of k-involutions was given in the case that $G = \mathrm{SL}(2,k)$ which does not depend on any of the results in [Hel00]. Similarly, this is done for $\mathrm{SL}(n,k)$ in [HWD04]. Using this characterization, the possible isomorphism classes for algebraically closed fields, the real numbers, the p-adic numbers, and the finite fields were classified. Analogous results for isomorphism classes of involutions of connected reductive algebraic groups can be found in [Hut14] for the exceptional group G_2 and in [BHJxx] for symplectic groups.

This concept can be generalized by considering order m k-automorphisms of G instead of k-involutions, which are of order two. We can then construct, in an analogous fashion, a generalized symmetric k-variety. To study these generalized symmetric k-varieties, first one needs a classification of the related order m k-automorphisms.

In this paper, we consider the order m k-automorphisms of $\mathrm{SL}(2,k)$ and characterize the isomorphy classes of these automorphisms. Throughout, we assume $m \geqslant 2$. In Section 2, we define some of the basic terminology that will be used and state previous results on the k-involutions of $\mathrm{SL}(2,k)$. In Section 3, we characterize the form that order m k-automorphisms of $\mathrm{SL}(2,k)$ take. In Section 4, we find simple conditions on matrices A and B, involving eigenvalues and the field that the entries of A and B lie in, that are equivalent to isomorphy between order m k-automorphisms Inn_A and Inn_B. In Section 5, we examine the occurrence of m-valid eigenpairs, which indicate an order m k-automorphism. In Sections 6, we

consider the number of isomorphy classes for a given field k, and order m. We conclude in Section 7 by examining the cases when $k = \overline{k}, \mathbb{R}$, $\mathbb{Q}$, or $\mathbb{F}_p$.

2. Preliminaries

We begin by defining some basic notation. Let k be a field of characteristic not two, $\bar{k}$ the algebraic closure of k,

$$\mathrm{M}(2,k) = \{2 \times 2\text{-matrices with entries in } k\},$$

$$\mathrm{GL}(2,k) = \{A \in \mathrm{M}(2,k) \mid \det(A) \neq 0\}$$

and

$$\mathrm{SL}(2,k) = \{A \in \mathrm{M}(2,k) \mid \det(A) = 1\}.$$

Let k^* denote the multiplicative group of nonzero elements of k, $(k^*)^2 = \{a^2 \mid a \in k^*\}$ denote the set of squares in k and $I \in \mathrm{M}(2,k)$ denote the identity matrix.

Definition 2.1. Let G be an algebraic groups defined over a field k. Let G_k be the k-rational points of G. Let $\mathrm{Aut}(G, G_k)$ denote the the set of k-automorphisms of G_k. That is, $\mathrm{Aut}(G, G_k)$ is the set of automorphisms of G which leave $G_k$invariant. We say $\vartheta \in \mathrm{Aut}(G, G_k)$ is a *k-involution* if $\vartheta^2 = \mathrm{id}$ but $\vartheta \neq \mathrm{id}$. A k-involution is a k-automorphism of order 2.

For $A \in G_k$, the map $\mathrm{Inn}_A(X) = A^{-1}XA$ is called an *inner k-automorphism of G_k*. We denote the set of such k-automorphisms by $\mathrm{Inn}(G_k)$. If $\mathrm{Inn}_A \in \mathrm{Inn}(G_k)$ is a k-involution, then we say that Inn_A is an *inner k-involution of G_k*.

Assume H is an algebraic group defined over k which contains G. Let H_k be the k-rational points of H. For $A \in H$, if the map $\mathrm{Inn}_A(X) = A^{-1}XA$ is such that $\mathrm{Inn}_A \in \mathrm{Aut}(G, G_k)$, then Inn_A is an *inner k-automorphism of G_k over H*. We denote the set of such k-automorphisms by $\mathrm{Inn}(H, G_k)$. If $\mathrm{Inn}_A \in \mathrm{Inn}(H, G_k)$ is a k-involution, then we say that Inn_A is an *inner k-involution of G_k over H*.

Suppose $\vartheta, \tau \in \mathrm{Aut}(G, G_K)$. Then ϑ is *isomorphic* to τ *over* H_k if there is φ in $\mathrm{Inn}(H_k)$ such that $\tau = \varphi^{-1}\vartheta\varphi$. Equivalently, we say that τ and ϑ are in the same *isomorphy class over H_k*.

For simplicity, we will refer to k-automorphisms simply as automorphisms for the remainder of this paper.

Definition 2.2. For a field k, we will refer to $k^*/(k^*)^2$ as the *square classes* of k.

For example, if $k = \overline{k}$, then $|k^*/(k^*)^2| = 1$ where 1 is a representative of this single square class. Further, $|\mathbb{R}^*/(\mathbb{R}^*)^2| = 2$ with representatives ± 1; the set $\{\mathbb{Q}^*/(\mathbb{Q}^*)^2\}$ is infinite with representatives ± 1 and all the prime numbers.

The following is the main result of [HW02].

Theorem 2.3. *Let k be a field of characteristic not two. Then* $\mathrm{SL}(2,k)$ *has exactly* $|k^*/(k^*)^2|$ *isomorphy classes of involutions.*

We will confirm this result in this paper, and see that the number of isomorphy classes of order m automorphisms where $m > 2$ does not depend on $|k^*/(k^*)^2|$.

3. Inner Automorphisms of SL$(2, k)$

Since the Dynkin diagram of SL$(2, k)$ has a trivial automorphism group, we know that all automorphisms of SL$(2, k)$ are of the form Inn_B for some $B \in \mathrm{GL}(2, \overline{k})$. We improve upon this fact in the following lemma.

Lemma 3.1. *If φ is an automorphisms of* SL$(2, k)$*, then $\varphi = \mathrm{Inn}_A$ for some $A \in \mathrm{SL}(2, k[\sqrt{\alpha}])$ where $\alpha \in k$, where each entry of A is a k-multiple of $\sqrt{\alpha}$.*

Proof. Let φ be an automorphism of SL$(2, k)$. We can write $\varphi = \mathrm{Inn}_B$ for some $B \in \mathrm{GL}(2, \overline{k})$. It follows from Lemma 4 of [HW02] that we can assume that $B \in \mathrm{GL}(2, k)$. Let $A = (\det(B))^{-\frac{1}{2}} B$ and $\alpha = \det(B)$. Note that $\alpha \in k$. By construction, we see that $\det(A) = 1$ and that the entries of A are k-multiples of $\sqrt{\alpha}$. □

We now consider a lemma which characterizes matrices in SL$(2, \overline{k})$.

Lemma 3.2. *Suppose $A \in \mathrm{SL}(2, \overline{k})$. Then A is of the form*

$$A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a + \lambda_1 + \lambda_2 \end{pmatrix}$$

or

$$A = \begin{pmatrix} \lambda_1 & 0 \\ c & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues of A, and $m_A(x)$ is the minimal polynomial of A.

Proof. If A is diagonal, then A is in the latter form where $c = 0$. We may assume A is not diagonal and write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We first assume that b is nonzero. We need only show that $c = -\frac{m_A(a)}{b}$ and $d = -a + \lambda_1 + \lambda_2$. The latter is clear since the the trace of A is $a + d = \lambda_1 + \lambda_2$. So we are only concerned with c.

Note that $m_A(x) = x^2 - \mathrm{trace}(A)x + \det(A) = x^2 - (\lambda_1 + \lambda_2)x + 1$ since A is a 2×2 matrix. Now, to find the value of c, recall that $ad - bc = 1$. Thus,

$$1 = a(-a + \lambda_1 + \lambda_2) - bc,$$

which implies that

$$bc = -a^2 + (\lambda_1 + \lambda_2)a - 1.$$

Since b is nonzero, we have that $c = -\frac{m_A(a)}{b}$.

We now suppose $b = 0$, then A is lower triangular and its diagonal entries must be its eigenvalues. Thus, $A = \begin{pmatrix} \lambda_1 & 0 \\ c & \lambda_2 \end{pmatrix}$. □

We can summarize the previous two lemmas into a characterization of the matrices $A \in \mathrm{SL}(2, k[\sqrt{\alpha}])$ that define order m automorphisms of SL$(2, k)$.

Theorem 3.3. *Suppose Inn_A is an order m automorphism of* SL$(2, k)$ *where $A \in \mathrm{SL}(2, k[\sqrt{\alpha}])$, $\alpha \in k$, and each entry of A is a k-multiple of $\sqrt{\alpha}$. Then,*

$$A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a + \lambda_1 + \lambda_2 \end{pmatrix}$$

or

$$A = \begin{pmatrix} \lambda_1 & 0 \\ c & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues of A, and $m_A(x)$ is the minimal polynomial of A.

4. Isomorphy Classes of Order m Automorphisms

In this section, we find conditions on the matrices A and B that determine whether or not Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$. We begin with a lemma that translates the isomorphy conditions from one about mappings to one about matrices.

Lemma 4.1. *Assume Inn_A and Inn_B are order m automorphisms of $\mathrm{SL}(2,k)$. Further, suppose A lies in $\mathrm{SL}(2,k[\sqrt{\alpha}])$ where each entry of A is a k-multiple of $\sqrt{\alpha}$, B lies in $\mathrm{SL}(2,k[\sqrt{\gamma}])$ where each entry of B is a k-multiple of $\sqrt{\gamma}$, where $\alpha,\gamma \in k$. Then Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$ if and only if there exists $Q \in \mathrm{GL}(2,k)$ such that $Q^{-1}AQ = B$ or $-B$.*

Proof. First assume there exists $Q \in \mathrm{GL}(2,k)$ such that $Q^{-1}AQ = B$ or $-B$. Then for all $U \in \mathrm{SL}(2,k)$, we have

$$\begin{aligned} \mathrm{Inn}_Q \mathrm{Inn}_A \mathrm{Inn}_{Q^{-1}}(U) &= Q^{-1}A^{-1}QUQ^{-1}AQ \\ &= (Q^{-1}AQ)^{-1}U(Q^{-1}AQ) \\ &= (\pm B)^{-1}U(\pm B) \\ &= B^{-1}UB \\ &= \mathrm{Inn}_B(U). \end{aligned}$$

So, $\mathrm{Inn}_Q \mathrm{Inn}_A \mathrm{Inn}_{Q^{-1}} = \mathrm{Inn}_B$ and Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$.

To prove the converse, we now assume that Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$. Then there exists $Q \in \mathrm{GL}(2,k)$ such that $\mathrm{Inn}_Q \mathrm{Inn}_A \mathrm{Inn}_{Q^{-1}} = \mathrm{Inn}_B$. We note that Inn_A and Inn_B are also automorphisms of $\mathrm{SL}(2,\overline{k})$. For all $U \in \mathrm{SL}(2,\overline{k})$, we have

$$Q^{-1}A^{-1}QUQ^{-1}AQ = B^{-1}UB,$$

which implies

$$BQ^{-1}A^{-1}QUQ^{-1}AQB^{-1} = U.$$

So, $Q^{-1}AQB^{-1}$ commutes with all elements of $\mathrm{SL}(2,\overline{k})$. We note that $Q^{-1}AQB^{-1} \in \mathrm{SL}(2,\overline{k})$, so $Q^{-1}AQB^{-1}$ must lie in the center of $\mathrm{SL}(2,\overline{k})$, which is $\{I,-I\}$. Thus $Q^{-1}AQ = B$ or $-B$. □

Note that Inn_A and Inn_B will be isomorphic only if A and B have entries in the same quadratic extension of k.

Lemma 4.2. *Assume Inn_A and Inn_B are order m automorphisms of $\mathrm{SL}(2,k)$, A lies in $\mathrm{SL}(2,k[\sqrt{\alpha}])$ where each entry of A is a k-multiple of $\sqrt{\alpha}$, and B lies in $\mathrm{SL}(2,k[\sqrt{\gamma}])$ where each entry of B is a k-multiple of $\sqrt{\gamma}$, where $\alpha,\gamma \in k$. If Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$, then $\gamma = c\alpha$. That is, α and γ lie in the same square class of k, and all of the entries of B are k-multiples of $\sqrt{\alpha}$.*

Proof. By Lemma 4.1, there exists $Q \in \mathrm{GL}(2,k)$ such that $Q^{-1}AQ = B$ or $-B$ and the result follows. $\square$

Using the previous theorem and lemmas, we can now characterize isomorphy classes of order m automorphisms of $\mathrm{SL}(2,k)$.

Theorem 4.3. *Suppose* Inn_A *and* Inn_B *are order* m *automorphisms of* $\mathrm{SL}(2,k)$ *where* A *and* $B \in \mathrm{SL}(2,k[\sqrt{\alpha}])$ *for some* $\alpha \in k$ *where each entry of* A *and* B *is a* k*-multiple of* $\sqrt{\alpha}$*.*

(a) If A *and* B *have the same eigenvalues,* λ_1 *and* λ_2*, then,* Inn_A *and* Inn_B *are isomorphic over* $\mathrm{GL}(2,k)$*.*
(b) If A *has eigenvalues* λ_1 *and* λ_2 *and* B *has eigenvalues* $-\lambda_1$ *and* $-\lambda_2$*, then* Inn_A *and* Inn_B *are isomorphic over* $\mathrm{GL}(2,k)$*.*
(c) If Inn_A *is isomorphic to* Inn_B *over* $\mathrm{GL}(2,k)$*, then* A *has the same eigenvalues as* B *or* $-B$*.*

Proof. (a) We consider two cases based on if λ_1 and λ_2 are k-multiples of $\sqrt{\alpha}$.

Case 1: If λ_1 and λ_2 are not k-multiples of $\sqrt{\alpha}$, then both A and B must not be lower triangular. We can assume

$$A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a+\lambda_1+\lambda_2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} c & d \\ -\frac{m_A(c)}{d} & -c+\lambda_1+\lambda_2 \end{pmatrix}.$$

Then for

$$Q_A = \begin{pmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{pmatrix} \in \mathrm{GL}(2,\overline{k}),$$

we have

$$Q_A^{-1}AQ_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Likewise, if we let

$$Q_B = \begin{pmatrix} d & d \\ \lambda_1 - c & \lambda_2 - c \end{pmatrix} \in \mathrm{GL}(2,\overline{k}),$$

it follows that

$$Q_B^{-1}BQ_B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Let

$$Q = Q_AQ_B^{-1} = \begin{pmatrix} \frac{b}{d} & 0 \\ \frac{c-a}{d} & 1 \end{pmatrix}.$$

Note that $Q^{-1}AQ = B$ and that $Q \in \mathrm{GL}(2,k)$. Using the result of Lemma 4.1, we have shown that Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$.

Case 2: Let λ_1 and λ_2 be k-multiples of $\sqrt{\alpha}$ and define $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. In this case, it is possible but not necessary that A and B are lower triangular.

If neither are triangular, then the argument from Case 1 shows that Inn_A and Inn_B are isomorphic over $\mathrm{GL}(2,k)$, as desired. Assume that A and B are lower triangular. We write

$$A = \begin{pmatrix} \lambda_1 & 0 \\ c & \lambda_2 \end{pmatrix}.$$

From Lemma 3.1, we know that λ_1, λ_2, and c are k-multiples of $\sqrt{\alpha}$. Let

$$Q_A = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{c} & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,k)$$

then

$$Q_A^{-1}AQ_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D.$$

Since A induces an order m automorphism of $\mathrm{SL}(2,k)$, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ must induce an order m automorphism of $\mathrm{SL}(2,k)$, Inn_D. We have shown that Inn_D is isomorphic over $\mathrm{GL}(2,k)$ to Inn_A by Lemma 4.1.

If B is lower triangular as well, then we can show that Inn_B is isomorphic to the automorphism induced by Inn_D. By transitivity of isomorphy, Inn_A is isomorphic to Inn_B over $\mathrm{GL}(2,k)$.

The only case left to consider is when A is not lower triangular, but B is lower triangular. It suffices to show that Inn_A is isomorphic over $\mathrm{GL}(2,k)$ to Inn_D, since we have already shown Inn_B is isomorphic to Inn_D. We again consider

$$A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a+\lambda_1+\lambda_2 \end{pmatrix} \in \mathrm{SL}(2,k[\sqrt{\alpha}])$$

and

$$Q_A = \begin{pmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{pmatrix} \in \mathrm{GL}(2,\overline{k}),$$

where

$$Q_A^{-1}AQ_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D.$$

Let $Q_2 = \sqrt{\alpha}Q_A$. Since all of the entries of Q_A are k-multiples of $\sqrt{\alpha}$, it follows that $Q_2 \in \mathrm{GL}(2,k)$. We can see that $Q_2^{-1}AQ_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$, and therefore Inn_A is isomorphic to Inn_D by Lemma 4.1.

(b) Suppose A has eigenvalues λ_1 and λ_2 and B has eigenvalues $-\lambda_1$ and $-\lambda_2$. Observe that A and $-B$ have the same eigenvalues. From the proof of (a), we know that Inn_A is isomorphic to Inn_{-B}. Since $\mathrm{Inn}_B = \mathrm{Inn}_{-B}$, we are done.

(c) Suppose Inn_A is isomorphic to Inn_B over $\mathrm{GL}(2,k)$. By Lemma 4.1, there exists $Q \in \mathrm{GL}(2,k)$ such that $Q^{-1}AQ = B$ or $-B$.

□

We summarize the results of this theorem in the following corollary.

Corollary 4.4. *Suppose* Inn_A *and* Inn_B *are order* m *automorphisms of* $\mathrm{SL}(2,k)$ *where* A *and* $B \in \mathrm{SL}(2,k[\sqrt{\alpha}])$ *for some* $\alpha \in k$ *and each entry of* A *and* B *is a* k*-multiple of* $\sqrt{\alpha}$*. Then* Inn_A *is isomorphic to* Inn_B *over* $\mathrm{GL}(2,k)$ *if and only if* A *has the same eigenvalues as* B *or* $-B$*.*

5. m-Valid Eigenpairs

In the previous section, we reduced the problem of isomorphy to a problem of eigenvalues and quadratic extensions. In this section, we consider the valid pairs of eigenvalues of a matrix A that could induce an automorphism of order m.

Definition 5.1. We call the pair $\lambda_1,\ \lambda_2 \in \overline{k}$ an *m-valid eigenpair* if Inn_A is an order m automorphism of $\mathrm{SL}(2,\overline{k})$ where $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \in \mathrm{SL}(2,\overline{k})$.

In the following two lemmas we characterize the matrices B where Inn_B acts as the identity on $\mathrm{SL}(2,k)$.

Lemma 5.2. *Suppose* Inn_B *for* $B \in \mathrm{GL}(n,\overline{k})$ *acts as the identity on* $\mathrm{SL}(2,k)$*. Then* $B = cI$ *for some* $c \in \overline{k}$*.*

Proof. This is Lemma 2 of [HW02]. $\square$

We can improve upon this statement since we can assume $B \in \mathrm{SL}(2,\overline{k})$. We can use this idea to characterize the matrices that induce order m automorphisms on $\mathrm{SL}(2,k)$.

Lemma 5.3. *(a) Suppose* Inn_B *for* $B \in \mathrm{SL}(2,\overline{k})$ *acts as the identity on* $\mathrm{SL}(2,k)$*. Then* $B = I$ *or* $B = -I$*.*

(b) Inn_A *is an order* m *automorphism of* $\mathrm{SL}(2,k)$ *if and only if* m *is the smallest integer such that* $A^m = I$ *or* $A^m = -I$*.*

Proof. (a) From Lemma 5.2, we have that $B = cI$ for some $c \in \overline{k}$. Since $B \in \mathrm{SL}(2,\overline{k})$, $\det(B) = 1 = c^2$, which means $c = \pm 1$.

(b) If m is the smallest integer such that $A^m = I$ or $A^m = -I$, then m is the smallest integer such that $\mathrm{Inn}_{A^m} = (\mathrm{Inn}_A)^m$ acts as the identity on $\mathrm{SL}(2,k)$, which means Inn_A is an order m automorphism of $\mathrm{SL}(2,k)$.

If Inn_A is an order m automorphism of $\mathrm{SL}(2,k)$, then Inn_{A^m} acts as the identity on $\mathrm{SL}(2,k)$. (a) implies that $A^m = I$ or $A^m = -I$. If there exists r such that $0 \leqslant r < m$ where $A^r = I$ or $A^r = -I$, then Inn_A is at most an order r automorphism of $\mathrm{SL}(2,k)$, which is a contradiction. Thus, m is the smallest integer such that $A^m = I$ or $A^m = -I$. $\square$

We can characterize the m-valid eigenpairs.

Theorem 5.4. λ_1 *and* λ_2 *are an* m*-valid eigenpair if and only if*

(a) λ_1 *is a primitive* $2m$*-th root of unity and* $\lambda_2 = \lambda_1^{2m-1}$*, or*

(b) m *is odd,* λ_1 *is a primitive* m*-th root of unity and* $\lambda_2 = \lambda_1^{m-1}$

Proof. Let $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. We begin by proving necessity, so assume that Inn_A is an order m automorphism of $\mathrm{SL}(2,\overline{k})$. We may assume that $A \in \mathrm{SL}(2,\overline{k})$ by

Lemma 3.1. By Lemma 5.3 (b), we know that m is the smallest integer such that $A^m = I$ or $A^m = -I$. There are two cases to consider.

First assume that m is the smallest integer that $A^m = -I$ and that $A^r \neq I$ when $0 \leqslant r \leqslant m$. Then λ_1 is a $2m$-th root of unity. Since $\det(A) = 1$, $\lambda_2 = \lambda_1^{2m-1}$.

Now assume that m is the smallest integer such that $A^m = I$ and that $A^r \neq -I$ when $0 \leqslant r \leqslant m$. Then λ_1 is an m-th root of unity. Since $\det(A) = 1$, then $\lambda_2 = \lambda_1^{m-1}$.

Now we prove the sufficiency of the conditions. In either case, $A \in \mathrm{SL}(2, \overline{k})$ follows from the construction of A. Let's first assume (a), then m is the smallest positive integer such that $\lambda_1^m = -1 = \lambda_2^m$, and $2m$ is the smallest integer such that $\lambda_1^{2m} = -1 = \lambda_2^{2m}$. Thus, m is the smallest integer such that $A^m = -I$ and $2m$ is the smallest integer such that $A^{2m} = I$. By Lemma 5.3 (b), Inn_A is an order m automorphism of $\mathrm{SL}(2, \overline{k})$.

Now assume the conditions of (b). Then m is the smallest integer such that $\lambda_1^m = 1 = \lambda_2^m$, and for every integer r where $0 \leqslant r < m$. We know that $\lambda_1^r \neq -1$, so m is the smallest integer such that $A^m = I$, and Lemma 5.3 (b) tells us that Inn_A is an order m automorphism of $\mathrm{SL}(2, \overline{k})$. □

Let φ denote Euler's φ-function. That is, for positive integer m, $\varphi(m)$ is the number of integers l such that $1 \leqslant l < m$ and $\gcd(l, m) = 1$.

Corollary 5.5. *For any given field k, there are $\varphi(m)$ m-valid eigenpairs.*

Proof. We consider separately the cases where m is odd and even. First, assume m is even. Write $m = 2^s t$ where s and t are integers and t is odd. If we include ordering, then there are $\varphi(2m)$ such pairs. This double counts the m-valid eigenpairs. Thus, the number of distinct m-valid eigenpairs is

$$\begin{aligned}\frac{\varphi(2m)}{2} &= \frac{\varphi(2^{s+1}t)}{2} \\ &= \frac{\varphi(2^{s+1})\varphi(t)}{2} \\ &= \frac{2^s\varphi(t)}{2} \\ &= 2^{s-1}\varphi(t) \\ &= \varphi(2^s)\varphi(t) \\ &= \varphi(2^s t) \\ &= \varphi(m).\end{aligned}$$

Now suppose m is odd. The eigenvalues may be primitive m-th or $2m$-th roots of unity. If we include ordering, there are $\varphi(m) + \varphi(2m)$ such pairs. Again, this double counts the m-valid eigenpairs. The number of distinct m-valid eigenpairs when m is odd is

$$\begin{aligned}\frac{\varphi(m) + \varphi(2m)}{2} &= \frac{\varphi(m) + \varphi(m)}{2} \\ &= \varphi(m).\end{aligned}$$

Regardless of the parity of m, there are always $\varphi(m)$ m-valid eigenpairs.

□

6. Number of Isomorphy Classes

Given a field k, not necessarily algebraically closed, we would like to know the number of the isomorphy classes of order m automorphisms of $\mathrm{SL}(2, k)$.

Definition 6.1. Let $C(m, k)$ denote the number of isomorphy classes of order m automorphisms of $\mathrm{SL}(2, k)$.

Theorem 6.2. *$C(m, k) = \frac{1}{2}\varphi(m)$ or 0 for $m > 2$, and $C(2, k) = |k^*/(k^*)^2|$.*

Proof. From Corollary 2 in [HW02], we know that $C(2, k) = |k^*/(k^*)^2|$. This is also clear from our results, since there is exactly one 2-valid eigenpair, consisting of the two roots of -1.

Now assume $m > 2$. We claim that each m-valid eigenpair induces either one or zero isomorphy classes. Recall that if Inn_A is an order m automorphism, then by Theorem 3.3 we may assume that λ is an m-th or $2m$-th primitive root of unity and

$$A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a + \lambda + \lambda^{-1} \end{pmatrix}$$

or

$$A = \begin{pmatrix} \lambda & 0 \\ c & \lambda^{-1} \end{pmatrix},$$

where $\det(A) = 1$ and the entries of A are in k, or are k-multiples of $\sqrt{\alpha}$ for some $\alpha \in k$. If $\lambda + \lambda^{-1}$ is nonzero, then $\lambda + \lambda^{-1}$ can lie in at most one square class of k. We need only show that $\lambda + \lambda^{-1} \neq 0$ when $m > 2$. If $\lambda + \lambda^{-1} = 0$, then we can rearrange this equation to get $\lambda^2 = -1$, which is the case only when $m = 2$.

In Corollary 5.5, we showed that there are always $\varphi(m)$ m-valid eigenpairs. It follows from Corollary 4.4 that if Inn_A and Inn_B are isomorphic where $A, B \in \mathrm{SL}(2, k[\sqrt{\alpha}])$, then A has the same eigenvalues as B or $-B$. So, Inn_A and Inn_{-A} are isomorphic. If A has eigenvalues λ and λ^{-1}, then $-A$ has eigenvalues $-\lambda$ and $-\lambda^{-1}$. Therefore, exactly two m-valid eigenpairs induce the same isomorphy class of order m automorphisms of $\mathrm{SL}(2, k)$, assuming the isomorphy classes exist. □

For the remainder of this section, we consider how many quadratic extensions of k can induce an order m automorphism of $\mathrm{SL}(2, k)$, specifically when $m > 2$.

Lemma 6.3. *Let k be a field, $\alpha \in k$, and suppose λ is an lth primitive root of unity.*

(a) If λ is a k-multiple of $\sqrt{\alpha}$, then so is λ^r for all odd integers r, and $\lambda^r \in k$ for all even integers r.
(b) If $\lambda + \lambda^{-1}$ is a k-multiple of $\sqrt{\alpha}$, then so is $\lambda^r + \lambda^{-r}$ for all odd integers r and $\lambda^r + \lambda^{-r} \in k$ for all even integers r.

Proof. The proof of (a) is clear. We probe (b) by induction. Let $r > 1$ be even and suppose $\lambda + \lambda^{-1}$ and $\lambda^{r-1} + \lambda^{-(r-1)}$ are k-multiples of $\sqrt{\alpha}$, and that $\lambda^{r-2} + \lambda^{-(r-2)} \in k$. Then

$$(\lambda + \lambda^{-1})(\lambda^{r-1} + \lambda^{-(r-1)}) = (\lambda^r + \lambda^{-r}) + (\lambda^{r-2} + \lambda^{-(r-2)}) \in k.$$

Thus, $\lambda^r + \lambda^{-r} \in k$.

Let $r > 1$ be odd and suppose $\lambda + \lambda^{-1}$ and $\lambda^{r-2} + \lambda^{-(r-2)}$ are k-multiples of $\sqrt{\alpha}$, and that $\lambda^{r-1} + \lambda^{-(r-1)} \in k$. Then an argument similar to the above shows that $\lambda^r + \lambda^{-r}$ is a k-multiple of $\sqrt{\alpha}$. □

From Theorem 6.2, if $m > 2$, then each m-valid eigenpair can induce at most one isomorphy class of order m automorphisms of $\mathrm{SL}(2,k)$. Paired Lemma 6.3, if $\mathrm{SL}(2,k)$ has an order m automorphism Inn_A, then the entries of matrices A that induce these automorphisms will have entries in k, or a single quadratic extension of k. This gives the following result.

Corollary 6.4. *If $m > 2$ and $\det(A) = 1 = \det(B)$, then it is not possible for Inn_A and Inn_B to be order m automorphisms and for A and B to have entries in distinct quadratic extensions of k.*

7. Examples

We now look at a few examples over different fields k.

Example 7.1 ($k = \overline{k}$)**.** Since all roots of unity will lie in k when k is algebraically closed, then every m-valid eigenpair, (λ_1, λ_2), will induce an order m automorphism of $\mathrm{SL}(2,k)$ of the form Inn_A where $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. The following results from Theorem 6.2:

Theorem 7.2. *$C(2,\overline{k}) = 1$ and $C(m,\overline{k}) = \frac{1}{2}\varphi(m)$ when $m > 2$.*

Example 7.3 ($k = \mathbb{R}$)**.** Let i denote the square root of -1 and λ be an lth primitive root of unity, where we assume $l = 2m$ or $l = m$ and m is odd. We know that (λ, λ^{l-1}) is an l-valid eigenpair by Theorem 5.4. For this eigenpair to induce an automorphism on $\mathrm{SL}(2,\mathbb{R})$, we need one of the following to be the case:

(a) $\lambda \in \mathbb{R}$;
(b) $\lambda = \gamma i$, for $\gamma \in \mathbb{R}$;
(c) $\lambda + \lambda^{l-1} \in \mathbb{R}$; or
(d) $\lambda + \lambda^{l-1} = \gamma i$, for $\gamma \in \mathbb{R}$.

These conditions follow since the entries of A must lie in $\mathbb{R}$ or be $\mathbb{R}$-multiples of i. (a) and (b) correspond to $A = \begin{pmatrix} \lambda & 0 \\ c & \lambda^{l-1} \end{pmatrix}$ inducing the automorphism Inn_A, and (c) and (d) correspond to $A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a + \lambda + \lambda^{l-1} \end{pmatrix}$ also inducing the automorphism Inn_A. Further, (a) and (c) correspond to the entries of A falling in $\mathbb{R}$, and (b) and (d) correspond to the entries of A being $\mathbb{R}$-multiples of i. Using De Moivre's formula, we can write

$$\lambda = \cos\left(\frac{2\pi r}{l}\right) + i \sin\left(\frac{2\pi r}{l}\right)$$

and

$$\lambda^{l-1} = \cos\left(\frac{2\pi r}{l}\right) - i \sin\left(\frac{2\pi r}{l}\right)$$

for some integer r where $0 < r < l$ and r is coprime to l. We can easily check to see when we have each of the four cases listed above.

(a) When is $\lambda \in \mathbb{R}$? If $\lambda \in \mathbb{R}$, then $\lambda^h \in \mathbb{R}$ for all integers h. So we may assume that $r = 1$. This will occur when $\sin\left(\frac{2\pi}{l}\right) = 0$. Thus, $l = 2$ and $\lambda = -1$. Since we are assuming $m \geqslant 2$, this cannot happen.

(b) When is $\lambda = \gamma i$, for $\gamma \in \mathbb{R}$? Similar to the previous case, we may assume that $r = 1$. Then $\lambda = \gamma i$, for $\gamma \in \mathbb{R}$ will occur when $\cos\left(\frac{2\pi}{l}\right) = 0$. This can happen only when $\frac{2\pi}{l} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, which yields $l = 4$ and $l = \frac{4}{3}$, respectively. The latter solution does not concern us, but the solution $l = 4$ occurs if $\lambda = i$. This happens when $m = 2$, and there is one 2-valid eigenpair, $(i, -i)$.

(c) When is $\lambda + \lambda^{l-1} \in \mathbb{R}$? Using De Moivre's formula, we see that

$$\begin{aligned}\lambda + \lambda^{l-1} &= \left(\cos\left(\frac{2\pi r}{l}\right) + i\sin\left(\frac{2\pi r}{l}\right)\right) + \left(\cos\left(\frac{2\pi r}{l}\right) - i\sin\left(\frac{2\pi r}{l}\right)\right)\\ &= 2\cos\left(\frac{2\pi r}{l}\right) \in \mathbb{R}.\end{aligned}$$

This is always the case.

(d) Based on the previous case, we see that $\lambda + \lambda^{l-1} = \gamma i$ for $\gamma \in \mathbb{R}$ is never the case.

If $m = 2$, then $l = 4$. There are two isomorphy classes of order 2 automorphisms: one where the matrix takes entries in $\mathbb{R}$ from (c), and one where the matrix has entries that are $\mathbb{R}$-multiples of i from case (b). Thus, $C(2,\mathbb{R}) = 2$, which agrees with the results in [HW02] and Theorem 6.2.

Suppose $m > 2$. Case (c) applies here. It follows that there are always mth and $2m$th primitive roots of unity. We have the following result.

Theorem 7.4. *If $m = 2$, then $C(2,\mathbb{R}) = 2$; if $m > 2$, then $C(m,\mathbb{R}) = \frac{1}{2}\varphi(m)$.*

Example 7.5 ($k = \mathbb{Q}$)**.** We know that $C(2,\mathbb{Q})$ is infinite. Consider the case where $m > 2$. As noted in the case where $k = \mathbb{R}$, if λ is an lth root of unity where $l = m$ or $2m$, then $\lambda + \lambda^{-1} = 2\cos\left(\frac{2\pi r}{l}\right)$. $\mathrm{SL}(2,\mathbb{Q})$ will have order m automorphisms if and only if $\cos\left(\frac{2\pi r}{l}\right)$ lies in $\mathbb{Q}$ or is a $\mathbb{Q}$ multiple of $\sqrt{p}$ for some prime p.

We first examine the case when $\cos\left(\frac{2\pi r}{l}\right)$ lies in $\mathbb{Q}$. By Niven's Theorem, Corollary 3.12 of [Niv56], $\cos x$ and $\frac{x}{\pi}$ are simultaneously rational only when $\cos x = 0, \pm\frac{1}{2}$, or ± 1. By Lemma 6.3, we may assume $r = 1$. Then $\cos\left(\frac{2\pi}{l}\right)$ is rational if and only if $l = 6, 4, 3, 2, \frac{3}{2}, \frac{4}{3}$, or $\frac{6}{5}$. Since l must be an integer, we need only consider $l = 6, 4, 3$, or 2. Since $m > 2$ we can further restrict our considerations to $l = 3$ or 6. Both of these correspond to order 3 automorphisms. There is $\frac{\varphi(3)}{2} = 1$ isomorphy class of order 3 automorphisms of $\mathrm{SL}(2,\mathbb{Q})$. If we let $l = 6$ and choose $a = b = 1$, then

$$A = \begin{pmatrix} a & b \\ -\frac{m_A(a)}{b} & -a + \lambda + \lambda^{l-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is a matrix that will induce an order 3 automorphism.

We now consider the case when $2\cos\left(\frac{2\pi r}{l}\right)$ is a $\mathbb{Q}$ multiple of $\sqrt{p}$ for some prime number p. Again, it is sufficient to consider the case where $r = 1$. We note the following lemma which is a part of Theorem 3.9 in [Niv56].

Lemma 7.6. *Let l be a positive integer. Then $2\cos\left(\frac{2\pi}{l}\right)$ is an algebraic integer which satisfies a minimal polynomial of degree $\frac{\varphi(l)}{2}$.*

Since we are interested in knowing when $2\cos\left(\frac{2\pi r}{l}\right) = \mu\sqrt{p}$ for some $\mu \in \mathbb{Q}$ and prime p, we need $2\cos\left(\frac{2\pi r}{l}\right)$ to satisfy a polynomial of the form $x^2 - \mu^2 p = 0$. A necessary condition for such l is that $\frac{\varphi(l)}{2} = 2$, or $\varphi(l) = 4$.

If $l = p^m$ for some prime p, then

$$4 = \varphi(p^m) = p^{m-1}(p-1).$$

Note that p and $p-1$ cannot both be even, so it must be the case that $p^{m-1} = 4$ and $p-1 = 1$, which means $l = 8$, or $p^{m-1} = 1$ and $p-1 = 4$, which means $l = 5$. If $l = p^m q^t$ for some distinct primes p and q, then

$$4 = \varphi(p^m q^t) = (p^m - p^{m-1})(q^t - q^{t-1}).$$

If $p^m - p^{m-1} = 2 = q^t - q^{t-1}$, then $p^m = 4$ and $q^t = 3$ which means $l = 12$. (Other primes and/or larger powers would not yield $\varphi(p^m) = 2$.) If $p^m - p^{m-1} = 4$ and $q^t - q^{t-1} = 1$, then $p^m = 8$ or 5, and $q^t = 2$. Since p and q are distinct, we have $l = 10$. If l is a multiple of three or more distinct primes, then $\varphi(l) > 4$. So, the only l for which $\varphi(l) = 4$ are $l = 5, 8, 10$ and 12. Note that

$$2\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{2},$$

$$2\cos\left(\frac{2\pi}{8}\right) = \sqrt{2},$$

$$2\cos\left(\frac{2\pi}{10}\right) = \frac{1+\sqrt{5}}{2},$$

and

$$2\cos\left(\frac{2\pi}{12}\right) = \sqrt{3}.$$

When $l = 8$ or 12, $2\cos\left(\frac{2\pi r}{l}\right)$ satisfies a polynomial of the form $x^2 - \mu^2 p = 0$, but no linear polynomial and for no other values of l. Thus, $\mathrm{SL}(2,\mathbb{Q})$ also has automorphisms of order 4 and 6.

Theorem 7.7. $\mathrm{SL}(2,\mathbb{Q})$ *only has finite order automorphisms of orders 1, 2, 3, 4, and 6. Further,* $C(2,\mathbb{Q})$ *is infinite, and* $C(3,\mathbb{Q}) = C(4,\mathbb{Q}) = C(6,\mathbb{Q}) = 1$.

Example 7.8 ($k = \mathbb{F}_q$, $q = p^r$, $p \neq 2$)**.** If $m = 2$, then $C(2,\mathbb{F}_q) = 2$. Again, assume $m > 2$. We need only determine when mth and $2m$th primitive roots of unity lie in $\mathbb{F}_q$ or are an $\mathbb{F}_q$-muliple of $\sqrt{\alpha}$ for some $\alpha \in \mathbb{F}_q$. We first consider the primitive roots which lie in $\mathbb{F}_q$. It is known that $\mathbb{F}_q \setminus \{0\}$ is a cyclic multiplicative group of order $q-1$, so it contains elements of orders $q-1$, and all of $(q-1)$'s divisors. Thus, $\mathbb{F}_q$ will contain all of the primitive roots of unity of orders $q-1$ and its divisors.

We now consider the primitive roots of unity which are $\mathbb{F}_q$ multiples of $\sqrt{\alpha}$ for some $\alpha \in \mathbb{F}_q$. Suppose $\lambda = \mu\sqrt{\alpha}$ where μ, $\alpha \in \mathbb{F}_q$. Note that

$$\lambda^{q-1} = \mu^{q-1}\alpha^{\frac{q-1}{2}} = \alpha^{\frac{q-1}{2}}.$$

It follows that $\lambda^{2(q-1)} = 1$. The maximal possible value l such that an lth primitive root of unity is an $\mathbb{F}_q$ multiple of $\sqrt{\alpha}$ for $\alpha \in \mathbb{F}_q$ is $2(q-1)$. To see that this maximal order of primitive roots of unity will always occur, suppose $\alpha \in \mathbb{F}_q$ is a $(q-1)$th primitive root of unity. Then $\sqrt{\alpha}$ is a $2(q-1)$th primitive root of unity. This, along with Theorem 6.2 proves the following result.

Theorem 7.9. *(a) If* $m = 2$, *then* $C(2,\mathbb{F}_q) = 2$.

(b) If $m > 2$ is even and $2m$ divides $2(q-1)$, or if m is odd and m (and $2m$) divides $q-1$, then $C(m,\mathbb{F}_q) = \frac{\varphi(m)}{2}$.
(c) In any other case, $C(m,\mathbb{F}_q) = 0$.

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